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Sensitivity of a Simple Earthquake Nucleation Model to Small Parameter Perturbation: Conditions for the Occurrence of Deterministic Chaos

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ABSTRACT

A new mechanism that may account for the onset of chaotic dynamics of earthquake faults is proposed and analyzed. The concept is to build on the Burridge-Knopoff model, which integrates the spring-block setup with the Dieterich-Ruina's rate- and state-dependent friction law to interpolate for the key aspects of earthquake episodes, including the seismic nucleation, fracture propagation and arrest, as well as the rupture healing. Results obtained indicate that determistic chaos occur in case frictional parameters exhibit small oscillations about their equilibrium values. Based on the construction of appropriate phase portraits, power spectra and the Lyapunov exponents it could be concluded that a single time-dependent parameter is sufficient for the chaotic behavior to emerge, while the fully developed chaos is found when two perturbed parameters are brought into play.

KEYWORDS

Spring-block model, External perturbation, Deterministics chaos

1. INTRODUCTION

The issues of seismogenesis, fracture propagation and healing, as well as the complex spatial, temporal and magnitude correlations linked to the observed earthquake patterns lie in the focus of an interdisciplinary research. Its advances are not important only for the scientific part on understanding the driven dissipative systems, but can also prove beneficial for their possible practical ramifications. A widely accepted definition states that earthquakes constitute stick-slip frictional instabilities recurring on preexisting faults driven by the slowly moving tectonic plates. In particular, each earthquake episode comprises a long-term phase of stress accumulation and a sudden jerky displacement accompanied by stress relaxation. Though there is no comprehensive framework that may account for all the related complex phenomena, it has come to light that the frictional laws, entering the earthquake models as "force" terms, are likely to be the crucial factor governing the statistical properties of earthquakes. Consistent with this, the physics of seismic phenomena is concerned both with the friction and fracture features relevant to the description on the microscopic level, and the application of statistical methods providing for the general event distributions on the macroscopic level. In terms of the former, the underlying friction laws can become complicated to say the least, involving not only the velocity and displacement dependencies, but also being contingent on the previous history ("prestress" from earlier cycles) and the "state" of the contact surface (interfacial layer). As for the event distributions, the long-term patterns of seismic activity can broadly be cast in two categories, one referring to periodic or quasiperiodic sequences of "characteristic" earthquakes (mainshocks) similar in spatial extent and moment, while the other encompasses the aperiodic activity patterns, where a wide range of event sizes can be expected. The rationale for the events obeying some systematic behavior often comes down to two scenarios: either there is a multicomponent fault with persistent segmentation, or the interaction of several faults may take place, such that the influence of all but one (master fault) can be neglected. Regarding the aperiodic behavior, the possible scenarios are more diverse. If the faults are made up of multiple segments which are only partially locked, the transfer of stress and its interplay with prestress may lead to local events where a single segment or a cluster of segments can take part, as well as the major events that break out along the entire seismogenic zone. The more likely setup for aperiodic behavior involves spatial variations (heterogeneity) of the friction laws, as patches of materials with different frictional properties may be embedded within an otherwise homogeneous fault. However, for the analysis pursued in the current paper, it is of greater importance that the aperiodic sequences of events can be linked to quite different dynamical backgrounds. Primarily, one may have low-dimensional data sets that are best accounted for by the deterministic chaos, or there may be some high-dimensional data sets, whose generation can naturally be attributed to stochastic (random) processes. One should caution for the instances where it may be difficult to distinguish between the stochastically perturbed deterministic chaos and the genuine random processes. Apart from the above opposition, a comparably novel explanation suggested for the aperiodic behavior, in particular the power-law magnitude-frequency distribution determined by the Gutenberg-Richter law, relies on the concept of self-organized criticality, by which the fault is assumed to consistently operate on a brink of a failure, such that the slow loading enables the system to self-organize into a dynamical critical state which lacks any characteristic time or length scale.

The paper is organized as follows. In section 2, we provide some background on the considered models, discussing the phenomena related to the original system and the introduced modifications. In section 3 we formally introduce the extended model, assuming that one or two system parameters are periodically perturbed. The section proceeds with the detailed local bifurcation analysis, supported by the phase portraits, power spectra and the calculation of the maximal Lyapunov exponent, the latter offering evidence for the chaotic behavior in the extended model. In section 4 we attempt to provide a seismological interpretation of the obtained results. Concluding remarks are given in section 5, where a brief summary of the results is followed by the discussion on possible implications to the earthquake phenomena, outlining the issues for further research.

2. BACKGROUND ON THE ORIGINAL MODEL AND ITS DERIVATIVES

In the present paper, we build on the Burridge-Knopoff (BK) model [1], which is counted among the canonical models of the fault dynamics. In the simplest configuration, it comprises a single block which is connected to the rigid loader plate via a stiff spring and lies on the frictional sliding surface analogous to the fault, see Fig. 1. The dynamics described by the usual one-dimensional version is likely to fit best the motion of large earthquakes that span the depth of the schizosphere and tend to propagate only in one dimension along the fault. The basic model and its various derivatives have proven capable of capturing the nucleation process accompanied by the low magnitude aseismic slips localized to the compact "seed" area of the rupture, kinematics of the rupture spreading and the dynamics of the fracture healing process. Nonetheless, in recent years the BK model has attracted much attention for efficiently reproducing the statistical properties of sequences of recurring earthquake events [2-5]. For this line of research, one of the key factors turns out to be the accurate representation of the involved characteristic spatial and temporal scales. In terms of the dynamics displayed, the focus in the BK model lies with the interplay of the driving and the friction forces which generates the stick-slip type of motion, paradigmatic for the propagation of earthquakes. In particular, the earthquake itself corresponds to the "slip" phase, whereas the "stick" stage reflects the interseismic period of the elastic strain accumulation. For the onset of a slip, the force acting on a block has to overcome the static friction with the surface. Nonetheless, in order for the model to exhibit a dynamical instability which allows for the fracture propagation, it is essential that the friction force possesses a frictional weakening property, meaning that the friction should become weaker as the block slides.

The formulation of a plausible friction law requires a deeper insight into the physics of an interfacial layer between the block and the rough surface, given that the interface bears the local pressures approaching the material yield strength, which renders its properties substantially different from those of the surrounding elastic medium. From the microscopic point of view, the key ingredient to friction is the notion of asperity micro-contacts, these being the discrete local junctions between the protrusions from the two surfaces brought into contact. The ensemble of asperities defines the genuine total area of contact which is much smaller than the nominal geometrical one. The onset and cessation of a slip are closely tied to the dynamics of the population of the asperities: the latter are rapidly detached just before the slip initiation, whereas the slip arrest promotes their renewal and strengthening. The slip itself is found to consist of two distinct phases, sharply distinguished by the characteristic time of duration and the slip rate. The initial, rapid slip phase, which commences immediately after the breaking of the asperities, is associated with an order of magnitude larger slip velocity than the proceeding slow slip phase.

These types of data, collected from a large number of experiments conducted under laboratory conditions within the velocity-step, slide-hold-slide and similar setups, have led to several constitutive friction laws, which have been successfully applied to explain for the various aspects of the stable and unstable sliding between the elastic solids. Such laws include the dependence of the friction strength on the slip velocity, as well as on a single or multiple evolving state variables which characterize the dynamics of the asperity contacts. In the present paper, we use the Dieterich-Ruina rate-and-state dependent friction law [1] which reads:

$$\mu = \mu 0 + A\ln\left(\frac{v}{v0}\right) + B\ln\left(\frac{v0\theta}{L}\right)$$
$$\dot{\theta} = 1 - \frac{v\theta}{L} \qquad (1)$$

where μ is the friction coefficient and μ_0 presents its value at some reference sliding velocity v_0. As for the parameters involved, A and B characterize the properties of the material, while L is a characteristic slip distance comparable to a typical asperity length. Finally, v refers to the slip velocity, whereas θ denotes the time-dependent state variable. Regarding the second equation, there are two points to be noted: first, θ has the dimension of time and second, it increases even if $v = 0$.

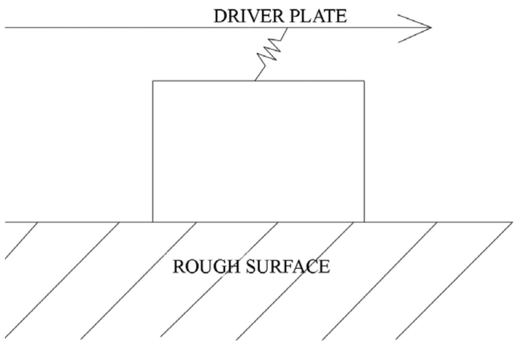

Figure 1: The Burridge-Knopoff model, represented by a slider-block coupled via a stiff spring to the loader plate.

Modification presented in this paper addresses the issue of the system's response to an external perturbation, this aimed at showing that even the small-amplitude influences are sufficient to profoundly change the original behavior, leading to the onset of chaos. Within this framework, the external perturbations are incorporated implicitly by assuming that they induce small oscillations about the equilibrium values of some of the system parameters. Such persistent time-dependent perturbations may be attributed to the Earth tides or the reservoir effects . However, note that the persistence of perturbations should be assessed in relative terms, meaning that even the impact of transient influences whose oscillation period is much shorter than the time they act on the system may still qualify for the provided description. In this context, one recalls the dynamical triggering models, which concern the possibility of earthquakes caused by the passage of seismic waves from the mainshock on some distant fault. In particular, it has been proposed that the stress pulse emitted by the mainshock may increase another fault's slip speed or enhance triggering by reducing the associated state variable.

One should note that analysis of the sensitivity of dynamical system to external perturbations is not limited to earthquake nucleation models, but also is the frequent research topic in other scientific areas [6].

Our numerical simulations of a spring-block model are based on the system of equations coupled with the Dieterich-Ruina rate- and state-dependent friction law are given by

$$\dot{\theta} = -\left(\frac{v}{L}\right)\left(\theta + B\log\left(\frac{v}{v_0}\right)\right)$$
$$\dot{u} = v - v_0 \qquad (2)$$
$$\dot{v} = \left(-\frac{1}{M}\right)\left(ku + \theta + A\log\left(\frac{v}{v_0}\right)\right)$$

where parameter M is the mass of the block, v_0 stands for the reference (normalizing) velocity, and the spring stiffness k accounts for the linear elastic properties of the bulk medium. Parameter L corresponds to the characteristic friction length, and can be interpreted as the sliding distance for the complete renewal of the population of asperity contacts. The parameters A and B are empirical constants, which depend on the properties of the materials involved. In particular, the parameter A measures the direct velocity dependence ("direct effect"), while $A–B$ is a measure of the steady-state velocity dependence. Note that the friction-related term is the only source of nonlinearity in the model.

For the improved insight into the arising phenomena, we adhere to a common practice of converting the initial system, such as (2), into the non-dimensional one. This is achieved by introducing the scaled variables θ', v', u' and t' defined as: $\theta=A\theta'$, $v=v_0v'$, $u=Lu'$, $t=(L/v_0)t'$. Substituting for the original set of variables θ, v, u and t, the system can finally be recast in the form:

$$\begin{aligned} \dot{\theta} &= -v(\theta+(1+\alpha)log(v)) \\ \dot{u} &= v-1 \\ \dot{v} &= -\eta^2\left[u+\left(\frac{1}{\beta}\right)(\theta+log(v))\right] \end{aligned} \tag{3}$$

where $\alpha=(B–A)/A$ measures the sensitivity of the velocity relaxation, $\beta=(kL)/A$ is the nondimensional spring constant, and $\eta=(k/M)^{1/2}(L/v_0)$ is the nondimensional characteristic frequency.

3. RESULTS

The original model (3) can be modified by assuming the time dependence of the parameters α and β in the following way:

$$\begin{aligned} \dot{\theta} &= -v\left[\vartheta+(1+\alpha(t))log\ v\right] \\ \dot{u} &= v-1 \\ \dot{v} &= -\gamma\eta^2\left[u+\left(\frac{1}{\beta}\right)\left(\theta+log\ v\right)\right] \end{aligned} \tag{4}$$

where $\alpha(t)$, $\beta(t)$ are the positive periodic functions of time:

$$\begin{aligned} \alpha(t) &= \alpha+\delta_\alpha sin(\omega_\alpha t) \\ \beta(t) &= \beta+\delta_\beta sin(\omega_\beta t) \end{aligned} \tag{5}$$

where δ_α, δ_β, ω_α, and ω_β present the constant oscillation amlitudes and the angular frequencies ($\delta_\alpha \leq \alpha$, $\delta_\beta \leq \beta$). Let us proceed with the bifurcation analysis of the model (4) if one or two of the underlying parameters are perturbed, while the remaining ones are held fixed at values admitting the fixed point (Fig. 2).

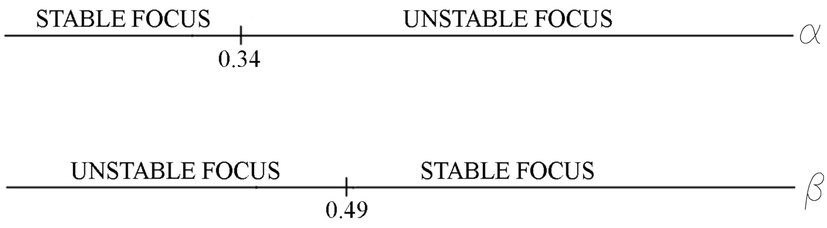

Figure 2: Bifurcations of the system (3) (or (4) if δi=0 is set) under variation of one of the parameters α, β and η. At each instance, the parameters held constant are awarded values that admit the fixed point, α= 0.34, β= 0.49 and η = 0.80

Furhter we exmaine the scenario if only a single parameter, α or β, undergoes small oscillations, while the other parameter is fixed. It turns out that one may observe chaotic behavior irrespective of which of the parameters is perturbed. The typical phase portraits and the power spectra for the corresponding time series are displayed in Fig. 3 (a-b). The emergence of chaotic behavior in these two instances is supported by calculating the maximal Lyapunov exponent, see Fig. 4, which indicates convergence to values above zero.

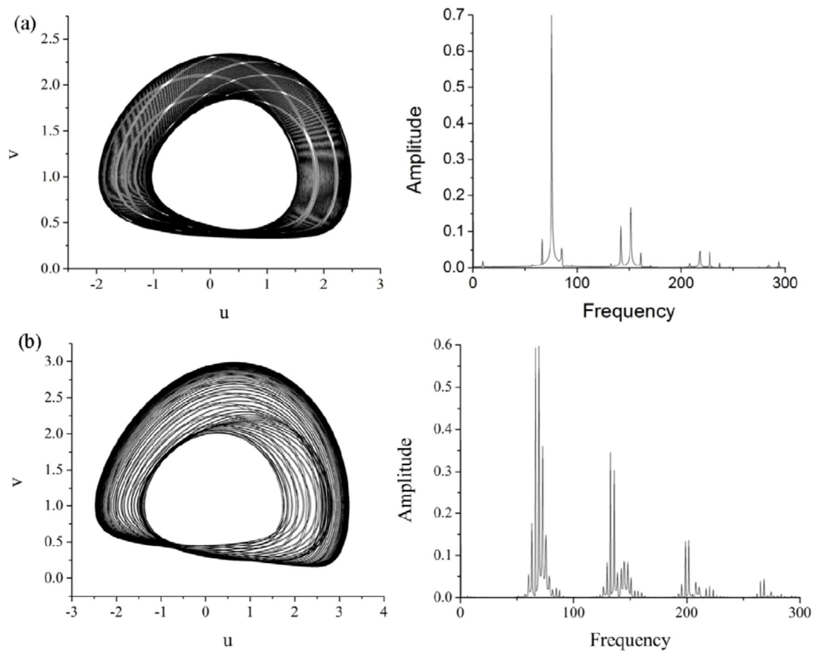

Figure 3: The left column shows the projections on the *(u,v)* plane of the typical orbits of the system (4), whereas the right column displays the corresponding power spectra obtained for the *v(t)* time series. Each row illustrates the system's behavior when the different parameter is periodically perturbed. The perturbation amplitudes are set to: (a) δ_α=0.4, ω_α=0.42 (b) δ_β=0.5, ω_β=0.42 while the equilibrium values of the parameters admit the limit cycle (α=0.4; β=0.5; η=0.8)

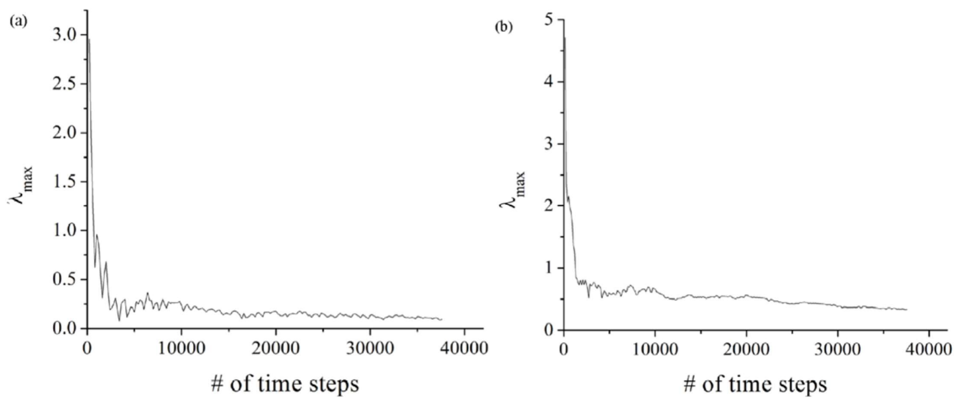

Figure 4: Calculation of the maximal Lyapunov exponent for simulated time series of the block's velocity for δ_α=0.4 in (a) and δ_β=0.5 in (b), respectively. The parameter values correspond to the plots displayed in Figures 4(a) and 4(b). It is demonstrated that the maximal Lyapunov exponents converge well to λ=0.095 in (a) and λ=0.336 in (b).

Further we proceed by examining what occurs if both α and β exhibit small oscillations. For such a scenario, it turns out that the fully developed chaos can indeed emerge, with an instance of the typical phase portrait shown in Fig. 5(a). It is interesting that the inference on the onset of chaos holds true even if the equilibrium parameter values are consistent with the fixed point of the unperturbed system. The conclusions on the character of the observed behavior rely on the power spectrum for the time series *v(t)*, provided in Fig. 5(b), and are further corroborated by the positive value of the maximal Lyapunov exponent, as indicated in Fig. 6. Note that the latter value is significantly larger than in the previous two cases where only a single parameter was perturbed.

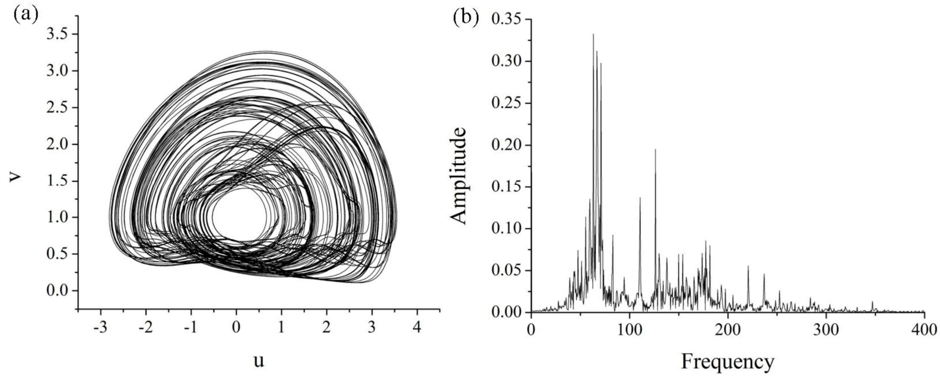

Figure 5: (a) Projection on the *(u,v)* plane of the typical chaotic orbit of the system (4). The results are obtained for the perturbation amplitudes δ_α=0.29 and δ_β=0.4, while the other parameters are set so that the unperturbed system would lie in the fixed point: α=0.29, β=0.5,η=0.8. (b) The power spectrum for the time series *v(t)*.

4. RELATION TO SEISMOLOGY

Though the studied spring-block model provides a greatly simplified picture of the seismic source, it may still shed some light on the mechanics of the fault system. Due to simplifying assumptions, the model is not intended to simulate the behaviour of any real fault, but can highlight the possible mechanisms contributing to the onset of complex behavior observed in the fault dynamics. In these terms, although the parameter values used in this paper are exclusively of theoretical character, without looking for the explicit relation with the observed data concerning the laboratory and natural fault zones, one may attempt to provide a qualitative interpretation of the obtained results. Our analysis involves three main parameters, namely *α, β* and *η*. Parameterη, representing the nondimensional characteristic frequency, incorporates the dependence on the parameters with constant values for the observed model: spring stiffness (*k*), mass of the block (*M*), critical slip distance (*L*) and the reference velocity (v_0). This is the reason whyη is held constant during the analysis. On the other hand, parameter *α* is defined by the ratio of parameters *B-A* and *A*. Parameters *A* and *B* stand for the material properties of the rock, which rely upon the rock type and temperature. These parameters also change during the slip: parameter *A* reflects the rise of the friction coefficient, when the block is subjected to a sudden velocity increase. For every successive slip phase, this parameter gains a new value, accounting for the different nature of the contact between the block and the rough surface. In addition, *A* enters the parameter *β*, defined as the nondimensional spring constant. Parameter *B-A* is directly connected with the amount of stress released during the ''slip'' phase. In our analysis, we first simulate this dependence by assuming that parameters *α* and *β* are periodically time-dependent, without relating to specific rock type and the temperature range. The results obtained indicate that for small perturbations near the fixed point, the system exhibits rich behavior, implying the sensitivity of the observed system on the material properties *A* and *B*. In other words, a transition to chaos is observed by simply changing the value of the parameters *A* and *B*. In an earthquake analogy to the single block, this suggests that the potential transitions to chaos are essentially controlled by the ratio of the parameters *(B–A)* and *A*, as well as the critical length *L* derived from the friction law, but are independent of the elastic feature of the medium surrounding the fault, which is idealized by the parameter *k*.

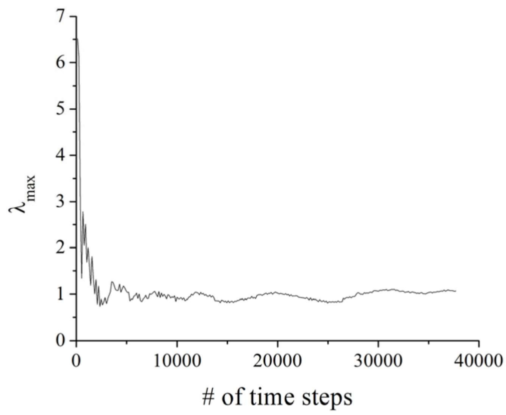

Figure 6: Calculation of the maximal Lyapunov exponent for the simulated time series of the block's velocity at δ_α=0.29 in (a) and δ_β=0.4 in (b), respectively. The other parameter values coincide with those in Fig. 5(a). The maximal Lyapunov exponent is seen to converge well to λ=1.068

5. CONCLUDING REMARKS

In principle, three dynamical paradigms are identified as likely to give rise to aperiodic behavior in the earthquake fault dynamics. These include the deterministic chaos, the stochastic processes and the self-organized citicality, none of which is at the current stage given advantage over the others. What makes the scenario involving deterministic chaos distinct is that it raises the possibility of the system being sensitive to small perturbations in the intrinsic or the external conditions. The purpose of this study has been to suggest the mechanisms that one may build into the minimal model of fault dynamics capable of exhibiting the chaotic behavior. The term „minimal" alludes to the point that even the fault with rather simplistic monoblock structure may still generate complex patterns of seismic activity. The adopted approach lies at variance with most of the models known so far to admit the aperiodic dynamics, given that they commonly invoke multicomponent faults composed of regions which are only partially locked or refer to spatially heterogeneous faults having sections with the different frictional laws intermixed. In particular, we have demonstrated that certain modifications, reflected in introducing the novel, small magnitude variables into the original BK monoblock model, may be sufficient to profoundly influence the system dynamics, eventually leading to the onset of the chaotic regime. In each instance, the observation of chaos is verified by determing the power spectra of the underlying *v(t)* time series, this corroborated by calculating the corresponding maximal Lyapunov exponent.

The focus of the present paper has been on the effects of external perturbations, which, due to low amplitude, may be incorporated implicitly into the model, having them induce small oscillations of the parameters α and β about their equilibrium values. In case of any single perturbed parameter, while the remaining subset is held fixed, one is unable to find the fully developed chaos. However, we have witnessed the latter once allowing for both α and β to be time-dependent. Note that the fully developed chaos is characterized by the continuous power spectrum of the appropriate time series and the leading Lyapunov exponent significantly above zero.

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